Hypothesis Testing

Perform the following steps in **any** test of hypotheses:

1. Determine the *null hypothesis* H_o.
2. Determine the *alternative hypothesis* H_a.
3. Choose the level of significance of the test ($\alpha-$ level).
4. Follow the appropriate decision rule (see below).

1 Single sample hypothesis tests

1.1 Hypothesis tests involving the mean μ of a population

1.1.1 σ known

Compute $z = \frac{\bar{x}-\mu_o}{\frac{\sigma}{\sqrt{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

- Two-sided test hypotheses
  $H_o : \mu = \mu_o$
  $H_a : \mu \neq \mu_o$
  Decision Rule: Reject if $z \leq -z_{\frac{\alpha}{2}}$[a], or if $z \geq z_{\frac{\alpha}{2}}$
- One-sided test of hypotheses
  - $H_o : \mu \geq \mu_o$
    $H_a : \mu < \mu_o$
    Decision Rule: Reject if $z \leq -z_\alpha$
  - $H_o : \mu \leq \mu_o$
    $H_a : \mu > \mu_o$
    Decision Rule: Reject if $z \geq z_\alpha$

1.1.2 σ unknown

Compute $t = \frac{\bar{x}-\mu_o}{\frac{s}{\sqrt{n}}}$, where $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

- Two-sided test hypotheses
  $H_o : \mu = \mu_o$
  $H_a : \mu \neq \mu_o$
  Decision Rule: Reject if $t \leq -t_{n-1;\frac{\alpha}{2}}$[b], or if $t \geq t_{n-1;\frac{\alpha}{2}}$
- One-sided test of hypotheses
  - $H_o : \mu \geq \mu_o$
    $H_a : \mu < \mu_o$
    Decision Rule: Reject if $t \leq -t_{n-1;\alpha}$
  - $H_o : \mu \leq \mu_o$
    $H_a : \mu > \mu_o$
    Decision Rule: Reject if $t \geq t_{n-1;\alpha}$

[a] $P(Z \geq z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$ (the tail of the normal density to the right of $\frac{\alpha}{2}$).
[b] $P(T \geq t_{n-1;\frac{\alpha}{2}}) = \frac{\alpha}{2}$ (the tail of the t density with $n-1$ degrees of freedom to the right of $\frac{\alpha}{2}$).

1.2 Hypothesis tests involving the proportion p of a characteristic of interest (successes) in a population

Compute $z = \frac{\hat{p}-p_o}{\sqrt{\frac{p_o(1-p_o)}{n}}}$, where $\hat{p} = \frac{x}{n}$ that is the number of "successes" x out of the number of "trials" (sample size) n.

- Two-sided test hypotheses
  $H_o : p = p_o$
  $H_a : p \neq p_o$
  Decision Rule: Reject if $z \leq -z_{\frac{\alpha}{2}}$, or if $z \geq z_{\frac{\alpha}{2}}$
- One-sided test of hypotheses
  - $H_o : p \geq p_o$
    $H_a : p < p_o$
    Decision Rule: Reject if $z \leq -z_{\alpha}$
  - $H_o : p \leq p_o$
    $H_a : p > p_o$
    Decision Rule: Reject if $z \geq z_{\alpha}$

Notice that when dealing with proportions, we do not have the case where σ is unknown. This is because the distribution of sample proportions is approximately normal with mean p and standard deviation $\sqrt{p(1-p)/n}$. By making the assumption (null hypothesis) that $p = p_o$ we also made the assumption that $\sigma = \sqrt{\frac{p_o(1-p_o)}{n}}$.

2 Two-sample Tests

When comparing the means of two populations, the following assumptions are made:

1. The populations are *normal*, with means μ_1 and μ_2 respectively, and
2. Both populations have the same (unknown) variance[c].

2.1 Hypothesis tests involving the means of the populations

When comparing two population means we are confronted with the following scenarios:

2.1.1 Independent samples

A typical example of such a case is a trial with two distinct samples of sizes n_1 and n_2 respectively, that are gathered from each population (e.g. a *control* and a *active treatment* group). These two groups are independent and similar in all other characteristics but the one of interest (usually a great deal of effort goes into the design of the clinical trial to achieve this), with means μ_1 and μ_2. We derive the following *pooled* estimate of the unknown common variance, using the samples that are collected from both populations:

$$\begin{aligned} s_p &= \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} \\ &= \sqrt{\frac{\sum_{i=1}^{n_1}(x_{i1}-\bar{x}_1)^2+\sum_{j=1}^{n_2}(x_{j2}-\bar{x}_2)^2}{n_1+n_2-2}} \end{aligned}$$

where $\bar{x}_1$ and $\bar{x}_2$ are the sample means of the two groups. The following tests are based on the statistic

$$t = \frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

- Two-sided test of hypotheses
  $H_o: \mu_1 = \mu_2$
  $H_a: \mu_1 \neq \mu_2$
  Decision Rule: Reject if $t \leq -t_{n_1+n_2-2;\frac{\alpha}{2}}$, or if $t \geq t_{n_1+n_2-2;\frac{\alpha}{2}}$. In this and the following cases, $t = \frac{\bar{x}_1-\bar{x}_2}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$, since $\mu_1-\mu_2 = 0$ according to the null hypothesis.
- One-sided test of hypotheses
  - $H_o: \mu_1 \geq \mu_2$
    $H_a: \mu_1 < \mu_2$
    Decision Rule: Reject if $t \leq -t_{n_1+n_2-2;\alpha}$
  - $H_o: \mu_1 \leq \mu_2$
    $H_a: \mu_1 > \mu_2$
    Decision Rule: Reject if $t \geq t_{n_1+n_2-2;\alpha}$

[c] Refer to section 11.2.2 in your textbook for the situation, where the two variances are unequal

2.1.2 Paired samples

This situation typically involves multiple measurements taken on the same subject. For example, a before treatment (baseline) measurement and a on treatment measurement. These measurements are not independent and the methods of the previous section do not apply. Instead, the tests are based on the *difference* $\bar{d} = \bar{X}_1 - \bar{X}_2$ of the sample means of the n (paired) measurements. Compute

$$t = \frac{\bar{d} - d_o}{\frac{s_d}{\sqrt{n}}}$$

where $s_d = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2}$, $d_i = (x_{i1} - x_{i2})$ and $\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$ is the sample mean difference. Note that in the case where the difference is zero under H_o, $t = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}}$. All tests proceed along the same lines as in subsection 1.1.2.

- Two-sided test hypotheses
  $H_o: d = 0$
  $H_a: d \neq 0$
  Decision Rule: Reject if $t \leq -t_{n-1;\frac{\alpha}{2}}$, or if $t \geq t_{n-1;\frac{\alpha}{2}}$
- One-sided test of hypotheses
  - $H_o: d \geq 0$
    $H_a: d < 0$
    Decision Rule: Reject if $t \leq -t_{n-1;\alpha}$
  - $H_o: d \leq 0$
    $H_a: d > 0$
    Decision Rule: Reject if $t \geq t_{n-1;\alpha}$

2.2 Comparison of two proportions

When comparing two proportions p_1 and p_2 from two independent samples, we proceed in a similar fashion as in subsection 2.1.1. If x_1 subjects out of n_1 in the first sample and x_2 out of n_2 in the second exhibit the characteristic of interest (death, development of AIDS, cancer remission, etc.) all tests are based on the sample proportions $\hat{p}_1 = \frac{x_1}{n_1}$ and $\hat{p}_2 = \frac{x_2}{n_2}$.
Compute $\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1+n_2} = \frac{x_1+x_2}{n_1+n_2}$. Now compute the statistic,

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

- Two-sided test of hypotheses
  $H_o: p_1 = p_2$
  $H_a: p_1 \neq p_2$
  Decision Rule: Reject if $z \leq -z_{\frac{\alpha}{2}}$, or if $z \geq z_{\frac{\alpha}{2}}$. In this and the following cases, $z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$,

  since $p_1 - p_2 = 0$ according to the null hypothesis.
- One-sided test of hypotheses
  - $H_o: p_1 \geq p_2$
    $H_a: p_1 < p_2$
    Decision Rule: Reject if $z \leq -z_\alpha$
  - $H_o: p_1 \leq p_2$
    $H_a: p_1 > p_2$
    Decision Rule: Reject if $z \geq z_\alpha$